

## SYMMETRIES ON TRANS-SASAKIAN SPACE FORMS

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ABSTRACT. In this article we have studied different types of symmetricity of trans-Sasakian space forms.

### 1. Introduction

In 1985, J.A. Oubiña [8] introduced a new class of almost contact manifold namely trans-Sasakian manifold of type  $(\alpha, \beta)$ . This class contains  $\alpha$ -Sasakian,  $\beta$ -Kenmotsu and cosymplectic manifolds. In particular when  $\alpha = 1$  and  $\beta = 0$  the manifolds are Sasakian manifolds which are analogues to Kähler manifolds. A Kähler manifold of constant holomorphic curvature is called a complex space form. Sasakian space forms are analogues to complex space forms. Many geometers [7, 1] studied the symmetric properties on Sasakian space forms. On the other hand a class of almost contact Riemannian manifolds abstracted by Kenmotsu [5] which are normal but not Sasakian are called Kenmotsu manifolds.  $\beta$ -Kenmotsu manifolds are the generalization of Kenmotsu manifolds. A Kenmotsu manifold with constant  $\varphi$ -holomorphic sectional curvature is called a Kenmotsu space form. In this article we first introduced the trans-Sasakian space form, and studied the several interesting symmetric properties as semi-symmetry, Ricci-semi-symmetry, pseudo-symmetry, Ricci-generalized-pseudo-symmetry, Weyl-projective-semi-symmetry and pseudo-projective-semi-symmetry on the trans-Sasakian space form.

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## 2. Preliminaries

Let  $M$  be a  $(2n+1)$  dimensional manifold and  $\varphi$ ,  $\xi$  and  $\eta$  be a tensor field of type  $(1,1)$ , a vector field, a 1 form on  $M$  respectively. If  $\varphi$ ,  $\xi$  and  $\eta$  satisfy the conditions

$$(2.1) \quad \eta(\xi) = 1 \quad \text{and} \quad \varphi^2 X = -X + \eta(X)\xi$$

for any vector field  $X$  on  $M$ , then  $M$  is said to have an **almost contact structure**  $(\varphi, \xi, \eta)$  and is called an **almost contact manifold**.

Using equation (2.1), for an almost contact structure  $(\varphi, \xi, \eta)$  one can prove the following properties:

- (i)  $\varphi(\xi) = 0$ ,
- (ii)  $\eta\circ\varphi = 0$ ,
- (iii)  $\text{rank}\varphi = 2n$ .

Every almost contact manifold  $M$  admits a Riemannian metric tensor field  $g$  such that

$$(2.2) \quad \eta(X) = g(X, \xi),$$

$$(2.3) \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y).$$

$$(2.4) \quad g(\varphi X, Y) = -g(X, \varphi Y).$$

The metric tensor field  $g$  called an **associated Riemannian metric tensor field** to the given almost contact structure  $(\varphi, \xi, \eta)$ . If  $M$  admits the structure  $(\varphi, \xi, \eta, g)$ ,  $g$  being an associated Riemannian metric tensor field of an almost contact structure  $(\varphi, \xi, \eta)$  then  $M$  is said to have an **almost contact metric structure**  $(\varphi, \xi, \eta, g)$  and is called an **almost contact metric manifold**.

For an  $(2n+1)$  dimensional almost contact manifold  $M$  with almost contact structure  $(\varphi, \xi, \eta)$ , we consider a product manifold  $M \times \mathbb{R}$ , where  $\mathbb{R}$  denotes a real line. Then a vector field on  $M \times \mathbb{R}$  is given by  $(X, f(d/dt))$ , where  $X$  is a vector field tangent to  $M$ ,  $t$  the coordinate of  $\mathbb{R}$  and  $f$  a function on  $M \times \mathbb{R}$ . We define a linear map  $J$  on the tangent space of  $M \times \mathbb{R}$  by

$$(2.5) \quad J(X, f \frac{d}{dt}) = (\varphi X - f\xi, \eta(X) \frac{d}{dt}).$$

Then we have  $J^2 = -I$  and hence  $J$  is an almost complex structure on  $M \times \mathbb{R}$ . The almost complex structure  $J$  is said to be integrable if its Nijenhuis torsion  $N$  vanishes, where

$$N(X, Y) = J^2[X, Y] + [JX, JY] - J[JX, Y] - J[X, JY].$$

If the almost complex structure  $J$  on  $M \times \mathbb{R}$  is integrable, we say that the almost contact structure  $(\varphi, \xi, \eta)$  is **normal**. A normal almost contact metric manifold is called **Sasakian manifold** [2]. The sectional curvature of the plane section spanned by the unit tangent vector field  $X$  orthogonal to  $\xi$  and  $\varphi X$  is called a  $\varphi$ -sectional curvature. If  $M$  has a constant  $\varphi$ -sectional curvature  $c$ , then  $M$  is called a **Sasakian space forms**.

Let  $(M, g)$  be an  $n$  dimensional Riemannian manifold  $n > 2$ , its curvature tensor defined by

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}.$$

Let  $T$  be  $(0, k)$  tensor, define a  $(0, 2 + k)$  tensor field  $R \cdot T$  by

$$\begin{aligned} (R \cdot T)(X_1, X_2, \dots, X_k, X, Y) &= R(X, Y)(T(X_1, X_2, \dots, X_k)) \\ &= -T(R(X, Y)X_1, X_2, \dots, X_k) \\ &\quad -T(X_1, R(X, Y)X_2, \dots, X_k) - \dots \\ &\quad -T(X_1, X_2, \dots, R(X, Y)X_k) \end{aligned}$$

one has

$$R(X, Y) \cdot T = \nabla_X (\nabla_Y T) - \nabla_Y (\nabla_X T) - \nabla_{[X, Y]} T.$$

When  $T = R$ , then we have a  $(0, 6)$  tensor  $R \cdot R$ .

The manifold  $(M, g)$  is called **semi-symmetric space** if

$$R \cdot R = 0.$$

and called **Ricci semi-symmetric space** if

$$R \cdot S = 0,$$

where  $S$  is the Ricci curvature tensor.

Also, we can determine a  $(0, k + 2)$  tensor field  $Q(A, T)$ , associated with any  $(0, k)$  tensor field  $T$  and any symmetric  $(0, 2)$  tensor field  $A$  by

$$\begin{aligned} (2.6) \quad Q(A, T)(X_1, X_2, \dots, X_k, X, Y) &= ((X \wedge_A Y) \cdot T)(X_1, X_2, \dots, X_k) \\ &= -T((X \wedge_A Y)X_1, X_2, \dots, X_k) \\ &\quad -T(X_1, (X \wedge_A Y)X_2, \dots, X_k) - \dots \\ &\quad -T(X_1, X_2, \dots, (X \wedge_A Y)X_k) \end{aligned}$$

where  $(X \wedge_A Y)$  is the endomorphism given by

$$(2.7) \quad (X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y.$$

Particulary, if we put  $A = g$  we get

$$(2.8) \quad (X \wedge_g Y)Z = g(Y, Z)X - g(X, Z)Y.$$

From now we will write  $(X \wedge_g Y)$  as  $(X \wedge Y)$ .

The **Weyl-projective curvature tensor**  $P$  on  $M$  is defined by

$$(2.9) \quad P(X, Y)Z = R(X, Y)Z - \frac{1}{2}[g(Y, Z)QX - g(X, Z)QY],$$

where  $Q$  is the Ricci operator  $Q$  defined by  $S(X, Y) = g(QX, Y)$ ,  $S$  being the Ricci curvature tensor. Another form of Weyl-projective curvature tensor is given by

$$(2.10) \quad \begin{aligned} P(X, Y)Z &= R(X, Y)Z - \frac{1}{2n}[S(Y, Z)X - g(X, Z)Y] \\ &= R(X, Y)Z - \frac{1}{2n}(X \wedge_S Y)Z. \end{aligned}$$

The manifold  $(M, g)$  is called **projectively-semi-symmetric space** if

$$R \cdot P = 0$$

and where  $P$  is the Weyl-projective-curvature tensor.

### 3. Trans Sasakian Manifold and Space form

Let  $(M, \varphi, \xi, \eta, g)$  be an almost contact metric manifold. If there are smooth functions  $\alpha, \beta$  on  $M$  satisfying

$$(\nabla\varphi)(X, Y) = \alpha[g(X, Y)\xi - \eta(Y)X] + \beta[g(\varphi X, Y)\xi - \eta(Y)\varphi X]$$

for all  $X, Y \in \mathfrak{X}(M)$ . Then the structure  $(\varphi, \xi, \eta, g, \alpha, \beta)$  is said to be a **trans-Sasakian structure** and the manifold  $(M, \varphi, \xi, \eta, g, \alpha, \beta)$  is said to be a **trans-Sasakian manifold** of type  $(\alpha, \beta)$ . Trans-Sasakian manifolds of type  $(0, 0)$ ,  $(\alpha, 0)$  and  $(0, \beta)$  are called cosymplectic,  $\alpha$ -Sasakian, and  $\beta$ -Kenmotsu manifolds respectively. Sasakian manifolds appear as examples of  $\alpha$ -Sasakian manifolds, with  $\alpha = 1$  and  $\beta = 0$  and Kenmotsu manifolds appear when  $\alpha = 0$  and  $\beta = 1$ . Marrero [6] has shown that a trans-Sasakian manifold of dimension  $\geq 5$  is either cosymplectic manifold, or  $\alpha$ -Sasakian manifold, or  $\beta$ -Kenmotsu manifold.

A trans-Sasakian manifold  $M^{2n+1}$  of constant  $\varphi$ -sectional curvature  $c$  is called a **trans-Sasakian space form** denoted by  $M^{2n+1}(c)$  and its curvature tensor is given by

$$(3.1) \quad \begin{aligned} R(X, Y)Z &= \frac{\alpha(c+3) + \beta(c-3)}{4} [g(Y, Z)X - g(X, Z)Y] \\ &+ \frac{\alpha(c-1) + \beta(c+1)}{4} \{[\eta(X)Y - \eta(Y)X]\eta(Z) + [g(X, Z)\eta(Y) \end{aligned}$$

$$-g(Y, Z)\eta(X)]\xi + g(\varphi Y, Z)\varphi X - g(\varphi X, Z)\varphi Y + 2g(X, \varphi Y)\varphi Z\}.$$

It can also be written by (2.2) and (2.8), as

$$(3.2) \quad R(X, Y)Z = (\alpha - \beta)(X \wedge Y)Z + \frac{\alpha(c-1) + \beta(c+1)}{4}\{(\varphi^2 X \wedge \varphi^2 Y)Z \\ + (\phi X \wedge \varphi Y)Z + 2g(X, \varphi Y)\varphi Z\}.$$

The Ricci tensor on trans-Sasakian space form defined by

$$(3.3) \quad S(X, Y) = \frac{1}{2}[c(n+1)(\alpha + \beta) + (3n-1)(\alpha - \beta)]g(X, Y) \\ - \frac{n+1}{2}[c(\alpha + \beta) - (\alpha - \beta)]\eta(X)\eta(Y).$$

It can also be written by (2.2), as

$$(3.4) \quad S(X, Y) = 2ng(X, Y) + \frac{n+1}{2}[c(\alpha + \beta) - (\alpha - \beta)]g(\varphi X, \varphi Y).$$

LEMMA 3.1. Let  $M^{2n+1}(c)$  be a trans-Sasakian space form and  $X, Y \in \mathfrak{X}(M)$ , then the following properties hold :

- (a)  $\varphi \cdot S = 0$ .
- (b)  $(X \wedge Y) \cdot S = 0$  iff  $c(\alpha + \beta) = \alpha - \beta$ .
- (c)  $(\varphi X \wedge \varphi Y) \cdot S = 0$ .
- (d)  $(\varphi^2 X \wedge \varphi^2 Y) \cdot S = 0$ .

*Proof.* (a) Since  $\varphi$  is a tensor field, we have

$$(\varphi \cdot S)(U, V) = -S(\varphi U, V) - S(U, \varphi V) \\ = -\frac{1}{2}[c(n+1)(\alpha + \beta) + (3n-1)(\alpha - \beta)]\{g(\varphi U, V) + g(U, \varphi V)\} \\ \text{[Using the property (ii)]} \\ = -\frac{1}{2}[c(n+1)(\alpha + \beta) + (3n-1)(\alpha - \beta)]\{g(\varphi U, V) - g(\varphi U, V)\} = 0 \\ \text{[by (2.4)]}$$

Thus  $(\varphi \cdot S)(U, V) = 0$  for any  $U, V \in \mathfrak{X}(M)$ .

(b) For any  $U, V \in \mathfrak{X}(M)$ , we have

$$((X \wedge Y) \cdot S)(U, V) = -S((X \wedge Y)U, V) - S(U, (X \wedge Y)V) \\ = -g(Y, U)S(X, V) + g(X, U)S(Y, V) \\ -g(Y, V)S(U, X) + g(X, V)S(U, Y) \quad \text{[by (2.8)]} \\ = -\frac{n+1}{2}[c(\alpha + \beta) - (\alpha - \beta)]\{-g(Y, U)\eta(X)\eta(V) + g(X, U)\eta(Y)\eta(V) \\ -g(Y, V)\eta(U)\eta(X) + g(X, V)\eta(U)\eta(Y)\} \quad \text{[by (3.3)]}$$

Since,  $\{-g(Y, U)\eta(X)\eta(V) + g(X, U)\eta(Y)\eta(V) - g(Y, V)\eta(U)\eta(X) + g(X, V)\eta(U)\eta(Y)\} \neq 0$  always and  $\alpha, \beta$  are nonzero functions, therefore

$$((X \wedge Y) \cdot S)(U, V) = 0 \text{ iff } c(\alpha + \beta) - (\alpha - \beta) = 0.$$

(c) For any  $U, V \in \mathfrak{X}(M)$ , we have

$$\begin{aligned} ((\varphi X \wedge \varphi Y) \cdot S)(U, V) &= -S((\varphi X \wedge \varphi Y)U, V) - S(U, (\varphi X \wedge \varphi Y)V) \\ &= -g(\varphi Y, U)S(\varphi X, V) + g(\varphi X, U)S(\varphi Y, V) \\ &\quad -g(\varphi Y, V)S(U, \varphi X) + g(\varphi X, V)S(U, \varphi Y) \\ &= \frac{1}{2} [c(n+1)(\alpha + \beta) + (3n-1)(\alpha - \beta)] \{-g(\varphi^2 Y, U)g(\varphi^2 X, V) \\ &\quad +g(\varphi^2 X, U)g(\varphi^2 Y, V) - g(\varphi^2 Y, V)g(U, \varphi^2 X) \\ &\quad +g(\varphi^2 X, V)g(U, \varphi^2 Y)\} \text{ [using (2.8) and property (ii)]} \\ &= 0. \end{aligned}$$

(d) Proof is similar to (c). □

**THEOREM 3.2.** *A trans-Sasakian space form  $M^{2n+1}(c)$  of type  $(\alpha, \beta)$  is Ricci-semi-symmetric if and only if  $c(\alpha + \beta) = \alpha - \beta$ .*

*Proof.* The curvature tensor is of the form

$$\begin{aligned} R(X, Y) &= (\alpha - \beta)(X \wedge Y) + \frac{\alpha(c-1) + \beta(c+1)}{4} \{(\varphi^2 X \wedge \varphi^2 Y) \\ &\quad + (\varphi X \wedge \varphi Y) + 2g(X, \varphi Y)\varphi\} \end{aligned}$$

So,

$$\begin{aligned} R(X, Y) \cdot S &= (\alpha - \beta)(X \wedge Y) \cdot S + \frac{\alpha(c-1) + \beta(c+1)}{4} \{(\varphi^2 X \wedge \varphi^2 Y) \cdot S \\ &\quad + (\varphi X \wedge \varphi Y) \cdot S + 2g(X, \varphi Y)\varphi \cdot S\} \end{aligned}$$

By the lemma-3.1, we have

$$R \cdot S = 0 \text{ if and only if } c(\alpha + \beta) - (\alpha - \beta) = 0. \quad \square$$

**LEMMA 3.3.** *Let  $M^{2n+1}(c)$  be a trans-Sasakian space form of type  $(\alpha, \beta)$  and  $X, Y \in \mathfrak{X}(M)$ , then the following properties hold :*

- (a)  $\varphi \cdot R = 0$ .
- (b)  $(\varphi X \wedge \varphi Y) \cdot R = -(X \wedge Y) \cdot R$
- (c)  $(X \wedge_S Y) \cdot R = 2n(X \wedge Y) \cdot R$

*Proof.* (a) For any  $X, Y, U, V \in \mathfrak{X}(M)$

$$\begin{aligned}(\varphi \cdot R)(X, Y, U, V) &= -R(\varphi X, Y, U, V) - R(X, \varphi Y, U, V) \\ &\quad -R(X, Y, \varphi U, V) - R(X, Y, U, \varphi V) \\ &= -g(R(\varphi X, Y)U, V) - g(R(X, \varphi Y)U, V) \\ &\quad -g(R(X, Y)\varphi U, V) - g(R(X, Y)U, \varphi V)\end{aligned}$$

Using property (ii) and after a long and straightforward computation we get

$$\begin{aligned}(\varphi \cdot R)(X, Y, U, V) &= \\ &\quad -\frac{\alpha(c-1) + \beta(c+1)}{4} [-g(\varphi Y, U)g(\varphi X, \varphi V) + g(\varphi X, \varphi U)g(\varphi Y, V) \\ &\quad -g(\varphi Y, \varphi U)g(\varphi X, V) + g(\varphi X, U)g(\varphi Y, \varphi V) + g(\varphi Y, \varphi U)g(\varphi X, V) \\ &\quad -g(\varphi X, \varphi U)g(\varphi Y, V) + g(\varphi Y, U)g(\varphi X, \varphi V) - g(\varphi X, U)g(\varphi Y, \varphi V)] = 0\end{aligned}$$

(b) For any  $X, Y, Z, U, V, W \in \mathfrak{X}(M)$ ,

$$\begin{aligned}((\varphi X \wedge \varphi Y) \cdot R)(Z, U, V, W) &= -R((\varphi X \wedge \varphi Y)Z, U, V, W) \\ &\quad -R(Z, (\varphi X \wedge \varphi Y)U, V, W) \\ &\quad -R(Z, U, (\varphi X \wedge \varphi Y)V, W) \\ &\quad -R(Z, U, V, (\varphi X \wedge \varphi Y)W) \\ &= -g(R[(\varphi X \wedge \varphi Y)Z, U]V, W) \\ &\quad -g(R[Z, (\varphi X \wedge \varphi Y)U]V, W) \\ &\quad -g(R[Z, U](\varphi X \wedge \varphi Y)V, W) \\ &\quad -g(R[Z, U]V, (\varphi X \wedge \varphi Y)W)\end{aligned}$$

Using property (ii) and (2.8) and after a long and straightforward computation we get the result.

(c) The Ricci curvature tensor can be written as

$$S(X, Y) = 2ng(X, Y) + \frac{n+1}{2} [c(\alpha + \beta) - (\alpha - \beta)] g(\varphi X, \varphi Y).$$

So, we have

$$S(Y, Z)X = 2ng(Y, Z)X + \frac{1}{2} [(n+1)(c-1)] g(\varphi Y, \varphi Z)X$$

and

$$S(X, Z)Y = 2ng(X, Z)Y + \frac{n+1}{2} [c(\alpha + \beta) - (\alpha - \beta)] g(\varphi X, \varphi Z)Y.$$

Thus,

$$(X \wedge_S Y)Z = S(Y, Z)X - S(X, Z)Y = 2n\{g(Y, Z)X - g(X, Z)Y\}$$

$$\begin{aligned}
& + \frac{n+1}{2} [c(\alpha + \beta) - (\alpha - \beta)] \{g(\varphi Y, \varphi Z)X - g(\varphi X, \varphi Z)Y\}. \\
= & 2n(X \wedge Y)Z + \frac{n+1}{2} [c(\alpha + \beta) - (\alpha - \beta)] \{g(\varphi Y, \varphi Z)X - g(\varphi X, \varphi Z)Y\}.
\end{aligned}$$

If we put  $Z = R$  (the Riemann curvature tensor), then

$$\begin{aligned}
(X \wedge_S Y) \cdot R = & 2n(X \wedge Y) \cdot R + \frac{n+1}{2} [c(\alpha + \beta) - (\alpha - \beta)] \{g(\varphi Y, \varphi \cdot R)X \\
& - g(\varphi X, \varphi \cdot R)Y\}.
\end{aligned}$$

From lemma-3.3(a)  $\varphi \cdot R = 0$ , therefore

$$(X \wedge_S Y) \cdot R = 2n(X \wedge Y) \cdot R.$$

□

DEFINITION 3.4. A Riemannian manifold  $(M, g)$ ,  $\dim M \geq 3$ , is said to be **pseudo-symmetric** (in the sense of R. Deszcz) if the  $(0, 6)$  tensor field  $R \cdot R$  and  $Q(g, R)$  on  $M$  are linearly dependent, i.e., if there exists a function  $\mathcal{L}_R: M \rightarrow \mathbb{R}$  such that

$$R \cdot R = \mathcal{L}_R Q(g, R)$$

holds on  $\mathcal{U}_R = \{x \in M \mid R - (\tau/n(n-1))G \neq 0\}$ , where  $\tau$  is the scalar curvature of  $M$  and  $G$  is the  $(0, 4)$  tensor field of  $M$  defined by

$$G(X_1, X_2, X_3, X_4) = g((X_1 \wedge X_2)X_3, X_4)$$

see [3].

THEOREM 3.5. Every trans-Sasakian space forms  $M^{2n+1}(c)$  is pseudo-symmetric, more precisely for every trans-Sasakian space forms:

$$R(X, Y) \cdot R = (\alpha - \beta)Q(g, R) = (\alpha - \beta)(X \wedge Y) \cdot R.$$

*Proof.* The curvature tensor is of the form

$$\begin{aligned}
R(X, Y) = & (\alpha - \beta)(X \wedge Y) + \frac{\alpha(c-1) + \beta(c+1)}{4} \{(\varphi^2 X \wedge \varphi^2 Y) \\
& + (\varphi X \wedge \varphi Y) + 2g(X, \varphi Y)\varphi\}
\end{aligned}$$

So,

$$\begin{aligned}
R(X, Y) \cdot R = & (\alpha - \beta)(X \wedge Y) \cdot R + \frac{\alpha(c-1) + \beta(c+1)}{4} \{(\varphi^2 X \wedge \varphi^2 Y) \cdot S \\
& + (\varphi X \wedge \varphi Y) \cdot R + 2g(X, \varphi Y)\varphi \cdot R\}
\end{aligned}$$

By the lemma-3.3 (a) and (b), we have

$$R(X, Y) \cdot R = (\alpha - \beta)(X \wedge Y) \cdot R = (\alpha - \beta)Q(g, R).$$

□



**COROLLARY 3.6.** A trans-Sasakian space forms  $M^{2n+1}(c)$  can not be semi-symmetric.

**DEFINITION 3.7.** In a Riemannian manifold  $(M, g)$ ,  $\dim M \geq 3$ , if the  $(0, 6)$  tensor field  $R \cdot R$  and  $Q(S, R)$  are linearly dependent, then the manifold is called **Ricci-generalized-pseudo-symmetric** [4]. That is equivalent to

$$R \cdot R = \mathcal{L}_S Q(S, R)$$

holding on  $\mathcal{U}_S = \{x \in M \mid Q(S, R) \neq 0\}$ , where  $\mathcal{L}_S$  is a function on  $\mathcal{U}_S$ .

**THEOREM 3.8.** A trans-Sasakian space forms  $M^{2n+1}(c)$  of type  $(\alpha, \beta)$  is Ricci-generalized-pseudo-symmetric.

*Proof.* By lemma-3.3 (c) and the result of Theorem 3.5

$$R(X, Y) \cdot R = (\alpha - \beta)(X \wedge Y) \cdot R = \frac{\alpha - \beta}{2n}(X \wedge_S Y) \cdot R = \mathcal{L}_S Q(S, R)$$

where  $\mathcal{L}_S = \frac{\alpha - \beta}{2n}$  is a function on  $M \supseteq \mathcal{U}_S$ . Hence the result.  $\square$

**LEMMA 3.9.** In a trans-Sasakian space form  $M^{2n+1}(c)$  the following are hold:

- (a)  $\varphi \cdot P = 0$ ,
- (b)  $(X \wedge Y) \cdot P = (X \wedge Y) \cdot R$

*Proof.* (a)

$$\begin{aligned} (\varphi \cdot P)(X, Y, U, V) &= -P(\varphi X, Y, U, V) - P(X, \varphi Y, U, V) \\ &\quad - P(X, Y, \varphi U, V) - P(X, Y, U, \varphi V) \\ &= -g(P(\varphi X, Y)U, V) - g(P(X, \varphi Y)U, V) \\ &\quad - g(P(X, Y)\varphi U, V) - g(P(X, Y)U, \varphi V) \\ &= \varphi \cdot R \quad [\text{by (2.4), (3.1) and (3.3)}] \\ &= 0. \quad [\text{by lemma-3.3(a)}] \end{aligned}$$

(b)

$$\begin{aligned} ((X \wedge Y) \cdot P)(Z, U, V, W) &= -P((X \wedge Y)Z, U, V, W) - P(Z, (X \wedge Y)U, V, W) \\ &\quad - P(Z, U, (X \wedge Y)V, W) - P(Z, U, V, (X \wedge Y)W) \\ &\quad \quad \quad [\text{by (2.8) and (2.10)}] \\ &= -R((X \wedge Y)Z, U, V, W) - R(Z, (X \wedge Y)U, V, W) \\ &\quad - R(Z, U, (X \wedge Y)V, W) - R(Z, U, V, (X \wedge Y)W) \\ &= ((X \wedge Y) \cdot R)(Z, U, V, W) \quad [\text{by (2.6)}] \end{aligned}$$

$\square$

**THEOREM 3.10.** *A trans-Sasakian space forms  $M^{2n+1}(c)$  of type  $(\alpha, \beta)$  is not projectively semi-symmetric.*

*Proof.* The curvature tensor of the form (3.2) is

$$R(X, Y) = (\alpha - \beta)(X \wedge Y) + \frac{\alpha(c-1) + \beta(c+1)}{4} \{(\varphi^2 X \wedge \varphi^2 Y) + (\varphi X \wedge \varphi Y) + 2g(X, \varphi Y)\varphi\}.$$

So,

$$\begin{aligned} R(X, Y) \cdot P &= (\alpha - \beta)(X \wedge Y) \cdot P + \frac{\alpha(c-1) + \beta(c+1)}{4} \{(\varphi^2 X \wedge \varphi^2 Y) \cdot P \\ &\quad + (\varphi X \wedge \varphi Y) \cdot P + 2g(X, \varphi Y)\varphi \cdot P\} \\ &= (\alpha - \beta)(X \wedge Y) \cdot R + \frac{\alpha(c-1) + \beta(c+1)}{4} \{(\varphi^2 X \wedge \varphi^2 Y) \cdot R \\ &\quad + (\varphi X \wedge \varphi Y) \cdot R\} \quad [\text{by lemma-3.9}] \\ &= (\alpha - \beta)(X \wedge Y) \cdot R + \frac{\alpha(c-1) + \beta(c+1)}{4} \{-(\varphi X \wedge \varphi Y) \cdot R \\ &\quad + (\varphi X \wedge \varphi Y) \cdot R\} \quad [\text{by lemma-3.3(b)}] \\ &= (\alpha - \beta)(X \wedge Y) \cdot R \neq 0 \quad [:\alpha \neq \beta]. \end{aligned}$$

□

**DEFINITION 3.11.** The **Pseudo projective curvature tensor**  $\bar{P}$  on a Riemannian manifold  $(M^{2n+1}, g)$  is defined as:

$$(3.5) \quad \bar{P}(X, Y)Z = aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y] - \frac{\tau}{2n+1} \left[ \frac{a}{2n} + b \right] [g(Y, Z)X - g(X, Z)Y].$$

By (2.8), we can write

$$(3.6) \quad \bar{P}(X, Y)Z = aR(X, Y)Z + b(X \wedge_S Y)Z - \frac{\tau}{2n+1} \left[ \frac{a}{2n} + b \right] (X \wedge Y)Z$$

where  $a$  and  $b$  are non-zero constants and  $\tau$  is the scalar curvature.

If  $a = 1$  and  $b = -\frac{1}{2n}$ , then (3.5) and (3.6) take the form

$$\bar{P}(X, Y)Z = P(X, Y)Z$$

where  $P$  is Projective curvature tensor. A Riemannian manifold is pseudo-projectively semi-symmetric if

$$R \cdot \bar{P} = 0.$$

- LEMMA 3.12.** (a)  $\varphi \cdot \bar{P} = 0$   
 (b)  $(X \wedge Y) \cdot \bar{P} = a(X \wedge Y) \cdot R$

Proof is similar to lemma-3.9.

**THEOREM 3.13.** *A trans-Sasakian space forms  $M^{2n+1}(c)$  of  $(\alpha, \beta)$  is not pseudo-projectively semi-symmetric.*

*Proof.* The curvature tensor of the form (3.2) is

$$R(X, Y) = (\alpha - \beta)(X \wedge Y) + \frac{\alpha(c-1) + \beta(c+1)}{4} \{(\varphi^2 X \wedge \varphi^2 Y) \\ + (\varphi X \wedge \varphi Y) + 2g(X, \varphi Y)\varphi\}.$$

Now,

$$\begin{aligned} R(X, Y) \cdot \bar{P} &= (\alpha - \beta)(X \wedge Y) \cdot \bar{P} + \frac{\alpha(c-1) + \beta(c+1)}{4} \{(\varphi^2 X \wedge \varphi^2 Y) \cdot \bar{P} \\ &\quad + (\varphi X \wedge \varphi Y) \cdot \bar{P} + 2g(X, \varphi Y)\varphi \cdot \bar{P}\} \\ &= a(\alpha - \beta)(X \wedge Y) \cdot R + \frac{\alpha(c-1) + \beta(c+1)}{4} \{a(\varphi^2 X \wedge \varphi^2 Y) \cdot R \\ &\quad + a(\varphi X \wedge \varphi Y) \cdot R\} \quad [\text{by lemma-3.12}] \\ &= a(\alpha - \beta)(X \wedge Y) \cdot R + \frac{\alpha(c-1) + \beta(c+1)}{4} \{-a(\varphi X \wedge \varphi Y) \cdot R \\ &\quad + a(\varphi X \wedge \varphi Y) \cdot R\} \quad [\text{by lemma-3.3(b)}] \\ &= a(\alpha - \beta)(X \wedge Y) \cdot R \neq 0 \quad [:\alpha \neq \beta \text{ and } a \neq 0]. \end{aligned}$$

□

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